

Towards a Realistic Equation of State of Strongly Interacting Matter

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Abstract

We consider a relativistic strongly interacting Bose gas. The interaction is manifested in the off-shellness of the equilibrium distribution. The equation of state that we obtain for such a gas has the properties of a realistic equation of state of strongly interacting matter, i.e., at low temperature it agrees with the

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one suggested by Shuryak for hadronic matter, while at high temperature it represents the equation of state of an ideal ultrarelativistic Stefan-Boltzmann gas, implying a phase transition to an effectively weakly interacting phase.

Key words: special relativity, strongly interacting matter, “realistic” equation of state, mass distribution

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1 Introduction

Quantum chromodynamics (QCD), the fundamental theory of strong interactions, can be perturbatively solved only in the region of asymptotic freedom, i.e., for large momenta of quarks and gluons [1]. At low momenta, quarks and gluons interact strongly and are confined inside hadrons. In this case, the expansion parameter of perturbation theory, the running coupling constant α_s , is of the order of unity, so that perturbative methods are not applicable and one has to use non-perturbative methods. One such method is the construction of effective theories to describe the behaviour of hadronic matter, such as the $\sigma - \omega$ model [2]. However, one of the most successful non-perturbative methods is the use of lattice gauge calculations [3], which are especially suitable for studying perturbative as well as non-perturbative effects in QCD. The presently available lattice data mostly concern simulations of $SU(N)$ pure gauge theory [4, 5], since the treatment of dynamical fermions on a lattice is difficult [3]. Moreover, lattice artefacts are believed to be well under control only in the pure gauge case [6]. A striking feature of lattice simulations of $SU(2)$ and $SU(3)$ pure gauge theory is a phase transition (of apparently first order for $SU(3)$ [4, 5, 7] and second order for $SU(2)$ theory [8]) from a phase of confined gluons (“glueballs”) to one of deconfined gluons (“gluon plasma”), leading to a sharp rise in the energy density of a gluon gas as a function of temperature at a phase transition temperature T_c [4, 5] (Figs. 1,2). In the $\sigma - \omega$ model, a similar phase transition (or, at least, a rapid increase of the energy density in a small temperature interval) appears at small net baryon densities [9] (Fig. 3). This is due to a strong enhancement of the scalar meson interaction at $T_c \simeq 200$ MeV, leading to a transition from a phase of massive baryons to a phase of massless baryons. Apparently, a phase transition to a weakly interacting phase seems to be a fundamental feature of strongly interacting matter¹ [10].

To understand the lattice data from a simple physical point of view, Rischke *et al.* [10, 12] have constructed a phenomenological model for the gluon plasma, in

¹It should be, however, mentioned that a phase transition in the $\sigma - \omega$ model may be an artefact of the approximations made in the calculation of the thermodynamic quantities in Fig. 3 (in the mean-field approximation) [11].

which gluons with large momenta are considered as an ideal gas with perturbative corrections of order $O(\alpha_s)$, while gluons with low momenta are subject to confining interactions and do not contribute to the energy spectrum of free gluons. The equation of state for this model, although it quantitatively reproduces the lattice data for the thermodynamic functions of $SU(3)$ pure gauge theory above the deconfinement transition temperature (Fig. 2), it has a rather complicated form and is therefore not suitable for practical use, e.g., in astrophysics (for description of stellar structure), or in cosmology (for treatment of a hadron-plasma phase transition in the early universe). In this paper, we suggest another equation of state, which reflects the main properties of strongly interacting matter (i.e., a phase transition to a weakly interacting phase) and agrees qualitatively with the lattice data for $SU(2)$ pure gauge theory *both* above and below the transition temperature, and has a much simpler form, as compared to that of ref. [10].

First, let us note that in strongly interacting matter, particles undergoing continual mutual interaction are necessarily off-shell. Therefore, the effect of strong interaction in such a system may be represented by the off-shellness of its particles. The equilibrium state of such a system should be characterized by a well-defined relativistic mass distribution around the on-shell value. Thus, instead of dealing with interaction explicitly, we reduce the problem to the description of the relativistic off-shell ensemble. The role of interaction then consists in determining the effective thermodynamic parameters governing the mass distribution in a strongly interacting system.

The physical framework for the description of a relativistic off-shell ensemble has been established by Horwitz and Piron [13] as a manifestly covariant relativistic dynamics whose consistent formulation is based on the ideas of Fock [14] and Stueckelberg [15], in which the four components of energy-momentum are considered as independent degrees of freedom, permitting fluctuations from the mass shell. In this framework, the dynamical evolution of a system of N particles, for the classical case, is governed by equations of motion that are of the form of Hamilton equations for the motion of N *events* which generate the space-time trajectories (particle world lines) as functions of a continuous Poincaré-invariant parameter τ [13, 15]. These events are characterized by their positions $q^\mu = (t, \mathbf{q})$ and energy-momenta $p^\mu = (E, \mathbf{p})$ in an $8N$ -dimensional phase-space. For the quantum case, the system is characterized by the wave function $\psi_\tau(q_1, q_2, \dots, q_N) \in L^2(R^{4N})$, with the measure $d^4q_1 d^4q_2 \dots d^4q_N \equiv d^{4N}q$, ($q_i \equiv q_i^\mu$; $\mu = 0, 1, 2, 3$; $i = 1, 2, \dots, N$), describing the distribution of events, which evolves with a generalized Schrödinger equation [13]. The collection of events (called “concatenation” [16]) along each world line corresponds to a *particle*, and hence, the evolution of the state of the N -event system describes, *a posteriori*, the history in space and time of an N -particle system.

For a system of N interacting events (and hence, particles) one takes [13] (we use the system of units in which $\hbar = c = k_B = 1$; we also use the metric $g^{\mu\nu} =$

$(-, +, +, +))$

$$K = \sum_i \frac{p_i^\mu p_{i\mu}}{2M} + V(q_1, q_2, \dots, q_N), \quad (1.1)$$

where M is a given fixed parameter (an intrinsic property of the particles), with the dimension of mass, taken to be the same for all the particles of the system. The Hamilton equations are

$$\begin{aligned} \frac{dq_i^\mu}{d\tau} &= \frac{\partial K}{\partial p_{i\mu}} = \frac{p_i^\mu}{M}, \\ \frac{dp_i^\mu}{d\tau} &= -\frac{\partial K}{\partial q_{i\mu}} = -\frac{\partial V}{\partial q_{i\mu}}. \end{aligned} \quad (1.2)$$

In the quantum theory, the generalized Schrödinger equation

$$i \frac{\partial}{\partial \tau} \psi_\tau(q_1, q_2, \dots, q_N) = K \psi_\tau(q_1, q_2, \dots, q_N) \quad (1.3)$$

describes the evolution of the N -body wave function $\psi_\tau(q_1, q_2, \dots, q_N)$.

In the present paper we restrict ourselves to a relativistic Bose gas, in order to compare the results with experimental data of pure gauge theory lattice simulations. We show that our results agree with those for $SU(2)$ pure gauge theory. It should be, however, noted that since the underlying theory is basically different from QCD, a comparison with the $SU(2)$ lattice data can only be qualitative. From this point of view, the similarity is remarkable.

2 Ideal relativistic Bose gas

Gibbs ensembles in a manifestly covariant relativistic classical and quantum mechanics were derived by Horwitz, Schieve and Piron [17]. To describe an ideal gas of events obeying Bose-Einstein statistics in the grand canonical ensemble, we use the expression for the number of events found in [17] (for our present purposes we assume no degeneracy),

$$N = V^{(4)} \sum_{k^\mu} n_{k^\mu} = V^{(4)} \sum_{k^\mu} \frac{1}{e^{(E - \mu - \mu_K \frac{m^2}{2M})/T} - 1}, \quad (2.1)$$

where $V^{(4)}$ is the system's four-volume and $m^2 \equiv -k^2 = -k^\mu k_\mu$ is the variable dynamical mass. Here, in addition to the usual chemical potential μ , there is the mass potential μ_K corresponding to the Lorentz scalar function $K(p, q)$ (Eq. (1.1)), here taken in the ideal gas limit, on the N -event relativistic phase space; in order to simplify subsequent considerations, we shall take it to be a fixed parameter (which determines an upper bound of the mass distribution in the ensemble we are studying,

as we shall see below). To ensure a positive-definite value for n_{k^μ} , the number of bosons with four-momentum k^μ , we require that

$$m - \mu - \mu_K \frac{m^2}{2M} \geq 0. \quad (2.2)$$

The discriminant of Eq. (2.2) must be nonnegative, which gives

$$\mu \leq \frac{M}{2\mu_K}. \quad (2.3)$$

For such μ , (2.2) has the solution

$$\frac{M}{\mu_K} \left(1 - \sqrt{1 - \frac{2\mu\mu_K}{M}} \right) \leq m \leq \frac{M}{\mu_K} \left(1 + \sqrt{1 - \frac{2\mu\mu_K}{M}} \right). \quad (2.4)$$

For small $\mu\mu_K/M$, the region (2.4) may be approximated by

$$\mu \leq m \leq \frac{2M}{\mu_K}. \quad (2.5)$$

One sees that μ_K plays a fundamental role in determining an upper bound of the mass spectrum, in addition to the usual lower bound $m \geq \mu$.

Replacing the sum over k^μ (2.1) by an integral, one obtains for the density of events per unit space-time volume $n \equiv N/V^{(4)}$ [18],

$$n = \frac{1}{4\pi^3} \int_{m_1}^{m_2} \frac{m^3 dm \sinh^2 \beta d\beta}{e^{(m \cosh \beta - \mu - \mu_K \frac{m^2}{2M})/T} - 1}, \quad (2.6)$$

where m_1 and m_2 are defined in Eq. (2.4), and we have used the parametrization [19]

$$\begin{aligned} p^0 &= m \cosh \beta, \\ p^1 &= m \sinh \beta \sin \theta \cos \phi, \\ p^2 &= m \sinh \beta \sin \theta \sin \phi, \\ p^3 &= m \sinh \beta \cos \theta, \end{aligned}$$

$$0 \leq \theta < \pi, \quad 0 \leq \phi < 2\pi, \quad -\infty < \beta < \infty.$$

In what follows we shall take $\mu \simeq 0$ (as for the case of the ensemble of gauge bosons). The integral (2.6) is calculated in ref. [20] (in the high-temperature Boltzmann approximation for the integrand):

$$n = \frac{T^4}{4\pi^3} \left[2 - x^2 K_2(x) \right], \quad x = \frac{2M}{T\mu_K} \equiv \frac{2m_c}{T}, \quad (2.7)$$

where $K_\nu(z)$ is the Bessel function of the third kind (imaginary argument), $m_c = M/\mu_K$ is the central value around which the mass of the particles are distributed, in

view of (2.4), and $2m_c$ is the upper limit of the mass distribution in our case of small μ , in view of (2.5). One calculates the characteristic averages [20] to be

$$\langle E \rangle = T \frac{8 - x^3 K_3(x)}{2 - x^2 K_2(x)}, \quad (2.8)$$

$$\langle E^2 \rangle = T^2 \frac{40 - x^4 K_4(x) + x^3 K_3(x)}{2 - x^2 K_2(x)}, \quad (2.9)$$

$$\langle m^2 \rangle = T^2 \frac{16 - x^4 K_4(x) + 4x^3 K_3(x)}{2 - x^2 K_2(x)}, \quad (2.10)$$

$$\langle \mathbf{p}^2 \rangle = 3T^2 \frac{8 - x^3 K_3(x)}{2 - x^2 K_2(x)} = 3T \langle E \rangle, \quad (2.11)$$

and obtains the following thermodynamic functions (the particle number density, pressure and energy density) ²:

$$N_0 = \langle J^0 \rangle = \frac{T^3}{\pi^2} \frac{8 - x^3 K_3(x)}{x^2}, \quad (2.12)$$

$$p = \frac{1}{3} \langle T^{ii} \rangle g_{ii} = \frac{T^4}{\pi^2} \frac{8 - x^3 K_3(x)}{x^2} = N_0 T, \quad (2.13)$$

$$\rho = \langle T^{00} \rangle = \frac{T^4}{\pi^2} \frac{40 - x^4 K_4(x) + x^3 K_3(x)}{x^2}, \quad (2.14)$$

where $\langle J^\mu \rangle$ and $\langle T^{\mu\nu} \rangle$ are the average *particle* four-current and energy-momentum tensor, respectively, given by [19]:

$$\langle J^\mu \rangle = \frac{T_{\Delta V}}{M} n \langle p^\mu \rangle, \quad \langle T^{\mu\nu} \rangle = \frac{T_{\Delta V}}{M} n \langle p^\mu p^\nu \rangle. \quad (2.15)$$

In (2.15), $T_{\Delta V}$ is the average passage interval in τ for the events which pass through the small (minimal typical) four-volume ΔV in the neighborhood of the R^4 -point; it is related to a width of the mass distribution around the central value, Δm , as follows [21]:

$$T_{\Delta V} \Delta m = 2\pi. \quad (2.16)$$

The expressions (2.13),(2.14) for p and ρ are obtained from Eq. (2.15) for $\langle T^{\mu\nu} \rangle$. They are, moreover, thermodynamically consistent. One may varify easily that the relation

$$\rho = T \frac{dp}{dT} - p \quad (2.17)$$

is satisfied [20].

²If we had not used the Boltzmann limit for the integrand in (2.6), one would obtain the factors $\zeta(3) \approx 1.202$ in Eq. (2.12) and $\zeta(4) = \pi^4/90 \approx 1.082$ in Eqs. (2.13),(2.14).

For low T , Eqs. (2.13),(2.14) reduce, through the asymptotic formula [23]

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \left(1 + \frac{4\nu^2 - 1}{8z} + \dots \right), \quad z \gg 1, \quad (2.18)$$

to

$$p = \frac{2T^6}{\pi^2 m_c^2}, \quad \rho = \frac{10T^6}{\pi^2 m_c^2} = 5p, \quad (2.19)$$

consistent with the “realistic” equation of state suggested by Shuryak for strongly interacting hadronic matter [22].

For high T , we use another asymptotic formula [24],

$$K_\nu(z) \sim \frac{1}{2} \Gamma(\nu) \left(\frac{z}{2} \right)^{-\nu} \left[1 - \frac{z^2}{4(\nu - 1)} + \dots \right], \quad z \ll 1, \quad (2.20)$$

and obtain

$$p = \frac{T^4}{\pi^2} \left(1 - \frac{m_c^2}{2T^2} \right) = p_{\text{SB}} \left(1 - \frac{m_c^2}{2T^2} \right), \quad (2.21)$$

$$\rho = \frac{T^4}{\pi^2} \left(3 - \frac{m_c^2}{2T^2} \right) = \rho_{\text{SB}} \left(1 - \frac{m_c^2}{6T^2} \right), \quad (2.22)$$

where $p_{\text{SB}} = T^4/\pi^2$ and $\rho_{\text{SB}} = 3p_{\text{SB}}$ are the pressure and energy density of an ideal ultrarelativistic Stefan-Boltzmann gas. Therefore, as $T \rightarrow \infty$, the thermodynamic functions of the relativistic Bose gas we are considering become asymptotically those of a Stefan-Boltzmann gas, implying a phase transition to an effectively weakly interacting phase, from the phase of strong interactions described by Eq. (2.19).

It follows from Eq. (2.10), via the asymptotic formulas (2.18),(2.20), that

$$\frac{\langle m^2 \rangle}{T^2} = \begin{cases} 8, & T \ll 2m_c, \\ 2m_c^2/T^2, & T \gg 2m_c. \end{cases} \quad (2.23)$$

The dependence of p/p_{SB} , ρ/ρ_{SB} and $\langle m^2 \rangle/T^2$ on temperature are shown in Figs. 4,5. At $T \simeq 0.2 m_c$ (corresponding to $z \simeq 0.1$ in Figs. 4,5), there is a smooth phase transition to a weakly interacting phase described by Eqs. (2.21),(2.22).

3 Concluding remarks

The manifestly covariant framework discussed in the present paper can be an effective tool in dealing with realistic physical systems. The equation of state (2.12)-(2.14) obtained in our work reflects the main properties of strongly interacting matter (i.e., a phase transition to a weakly interacting phase), and agrees qualitatively with the lattice data for $SU(2)$ pure gauge theory.

The question naturally arises of why the $SU(2)$ lattice data [8] appear to contain a second order phase transition but the $SU(3)$ data [4, 5] appear to contain first order one. There is no *a priori* difference between $SU(2)$ and $SU(3)$ pure gauge theories from a statistical mechanical point of view. As shown in ref. [4], $SU(3)$ pure gauge theory simulations on $24^3 \times N_T$ lattices indicate a rapid rise in $\rho + p$ as a function of temperature which takes place in the case of $N_T = 4$, reflecting a sharp first order phase transition. This rise is broadened considerably in the case of $N_T = 6$, with the slope of the curve diminished by almost a factor 3, indicating a smoother transition in this case. As remarked by the authors, this softening of the structure of the transition as N_T is increased may well continue as the continuum limit is approached. Thus, in ref. [4] the apparent first order nature of the transition in the case of $SU(3)$ pure gauge theory has been called in question. Moreover, there are indications from lattice QCD calculations that when fermions are included, the phase transition may be of second or higher order [25]. Altogether, these observations suggest that a realistic equation of state of strongly interacting hadronic matter should be expected to contain a second or higher order phase transition, as reflected by the equation of state obtained in our work.

As remarked by Ornik and Weiner [26], a *single* equation of state which, at high temperature, describes a quark-gluon phase and, at low temperature, a hadronic phase and which contains a phase transition of either first or higher order provides a more satisfactory theoretical description than one in which each phase is described by a different equation of state. In this way, the equation of state that we obtained can be considered as a candidate for such a realistic equation of state of strongly interacting matter, in contrast to the equations of state of, e.g., Rischke *et al.* [10] and Shuryak [22] each of which describes just one of the phases (above and below the transition, respectively). The introduction of the quark degrees of freedom in this equation of state, as well as taking into account an effective interaction potential in an explicit form in the strongly interacting phase, and perturbative corrections in the weakly interacting phase, should enable one to derive an equation of state which we expect to be more accurate for the description of the phenomena taking place in strongly interacting hadronic matter. The derivation of such an equation of state and its possible implications in astrophysics and cosmology are now being worked out by the authors.

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FIGURE CAPTIONS

Fig. 1. The energy density of $SU(2)$ pure gauge theory on a $10^3 \times 3$ lattice as a function of T/Λ_L . Taken from ref. [8].

Fig. 2. Fit of the thermodynamic functions, according to the equation of state of ref. [10], to the lattice data for $SU(3)$ pure gauge theory. Taken from ref. [10].

Fig. 3. Phase transition in the $\sigma - \omega$ model. Taken from ref. [10].

Fig. 4. The pressure and energy density, according to the equation of state (2.13),(2.14), as functions of $z = T/2m_c$.

Fig. 5. The average mass squared, according to Eq. (2.10), as a function of $z = T/2m_c$.